Cheating at your A-levels

(using theoretical computer science)

Olly Britton

A game

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2 \cdot \frac{15}{2} = 3 \cdot 5$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$3 \cdot 5$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$3 \cdot 5 \cdot \frac{55}{1} = 3 \cdot 5^2 \cdot 11$$

A game

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$$3 \cdot 5^2 \cdot 11 \cdot \frac{29}{33} = 5^2 \cdot 29$$

A game

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...after many more steps...

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2^2 \cdot 17$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2^2 \cdot 17 \cdot \frac{1}{17} = 2^2$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

...after many more steps...

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2^3 \cdot 17 \cdot \frac{1}{17} = 2^3$$

A game

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$$2^5 \cdot 17 \cdot \frac{1}{17} = 2^5$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2^7 \cdot 17 \cdot \frac{1}{17} = 2^7$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2^{11} \cdot 17 \cdot \frac{1}{17} = 2^{11}$$

A game

$$\underset{A}{\frac{17}{91}}, \underset{B}{\frac{78}{85}}, \underset{C}{\frac{19}{51}}, \underset{D}{\frac{23}{38}}, \underset{E}{\frac{29}{33}}, \underset{F}{\frac{77}{29}}, \underset{G}{\frac{95}{23}}, \underset{H}{\frac{77}{19}}, \underset{I}{\frac{1}{17}}, \underset{J}{\frac{11}{13}}, \underset{K}{\frac{13}{11}}, \underset{L}{\frac{15}{2}}, \underset{M}{\frac{1}{7}}, \underset{N}{\frac{55}{1}}$$

$$2, 2^2, 2^3, 2^5, 2^7, 2^{11}, \ldots$$

A game

$$2, 3, 5, 7, 11, \ldots$$

Theorem (Conway '87)
When PRIMEGAME:

$$\frac{17}{91}, \frac{78}{85}, \frac{19}{51}, \frac{23}{38}, \frac{29}{33}, \frac{77}{29}, \frac{95}{23}, \frac{77}{19}, \frac{1}{17}, \frac{11}{13}, \frac{13}{11}, \frac{15}{2}, \frac{1}{7}, \frac{55}{1}$$

is started at 2, the other powers of 2 that appear, namely,

$$2^2, 2^3, 2^5, 2^7, 2^{11}, 2^{13}, 2^{17}, 2^{19}, 2^{23}, 2^{29}, \dots$$

are precisely those whose indices are the prime numbers, in order of magnitude.

Fraction games in general

Definition
To play the *fraction game* corresponding to a given list

$$f_1, f_2, \ldots, f_k$$

of fractions and starting integer N, you repeatedly multiply the integer you have at any state (initially N) by the earliest f_i in the list for which the answer is integral. Whenever there is no such f_i, the game stops.

Theorem (Conway '87)
Define $f_c(n) = m$ if POLYGAME:

$$\frac{583}{559}, \frac{629}{551}, \frac{437}{527}, \frac{82}{517}, \frac{615}{329}, \frac{371}{129}, \frac{1}{115}, \frac{53}{86}, \frac{43}{53}, \frac{23}{47}, \frac{341}{46},$$

$$\frac{41}{43}, \frac{47}{41}, \frac{29}{37}, \frac{37}{31}, \frac{37}{29}, \frac{299}{23}, \frac{47}{15}, \frac{161}{19}, \frac{527}{7}, \frac{159}{17}, \frac{1}{13}, \frac{1}{3}$$

when started at $c2^{2^n}$, stops at 2^{2^m} and otherwise leave $f_c(n)$ undefined. Then every computable function appears among $f_0, f_1, f_2, \ldots$.

i.e. for any computer program, you can find a fraction game that simulates it

Addition program

$$\frac{3}{2}$$

$$2^a 3^b \mapsto 3^{a+b}$$

Multiplication program

$$\frac{455}{33}, \frac{11}{13}, \frac{1}{11}, \frac{3}{7}, \frac{11}{2}, \frac{1}{3}$$

$$2^a 3^b \mapsto 5^{ab}$$

Calculating π

$$\frac{365}{46}, \frac{29}{161}, \frac{79}{575}, \frac{679}{451}, \frac{3159}{413}, \frac{83}{407}, \frac{473}{371}, \frac{638}{355}, \frac{434}{335}, \frac{89}{235}, \frac{17}{209}, \frac{79}{122},$$
$$\frac{31}{183}, \frac{41}{115}, \frac{517}{89}, \frac{111}{83}, \frac{305}{79}, \frac{23}{73}, \frac{73}{71}, \frac{61}{67}, \frac{37}{61}, \frac{19}{59}, \frac{89}{57}, \frac{41}{53}, \frac{833}{47}, \frac{53}{43},$$
$$\frac{86}{41}, \frac{13}{38}, \frac{23}{37}, \frac{67}{31}, \frac{71}{29}, \frac{83}{19}, \frac{475}{17}, \frac{59}{13}, \frac{41}{291}, \frac{1}{7}, \frac{1}{11}, \frac{1}{1024}, \frac{1}{97}, \frac{89}{1}$$

$$2^n \mapsto 2^{\pi(n)}$$

where $\pi(n)$ is the n-th digit of π.

Cheating in your exams?

2024 JCQ instructions for conducting examinations

Calculators must not:

- *be designed to offer any of these facilities:*
    - *language translators;*
    - *symbolic algebra manipulation;*
    - *symbolic differentiation or integration;*

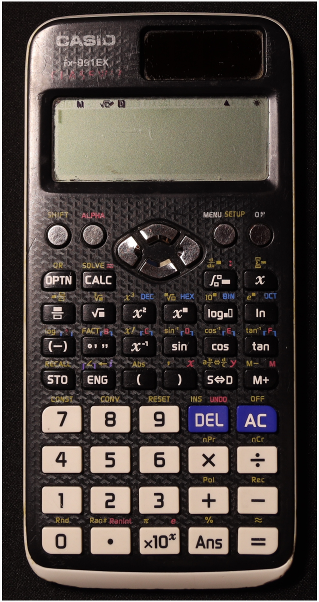
2024 JCQ instructions for conducting examinations

No simulating arbitrary Turing machines in an exam!

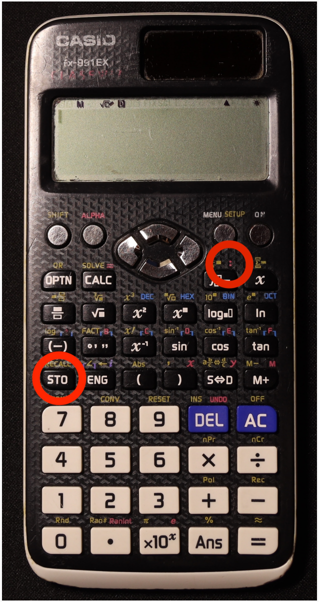
Cheating in your exams

Fraction games are actually simple enough that your calculator can simulate them.

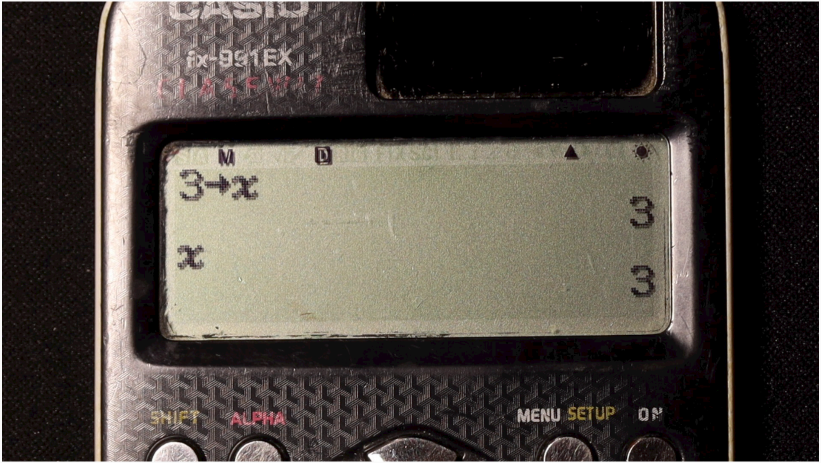
Casio fx-991EX



Casio fx-991EX

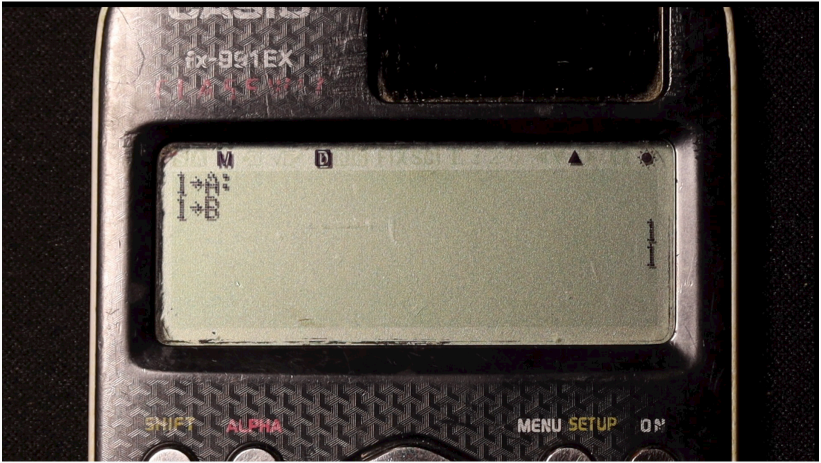


Variable assignment

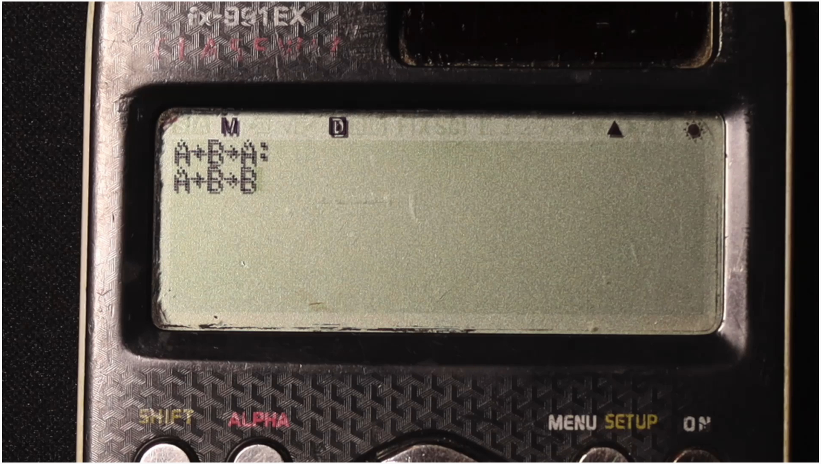


Separating expressions with a colon

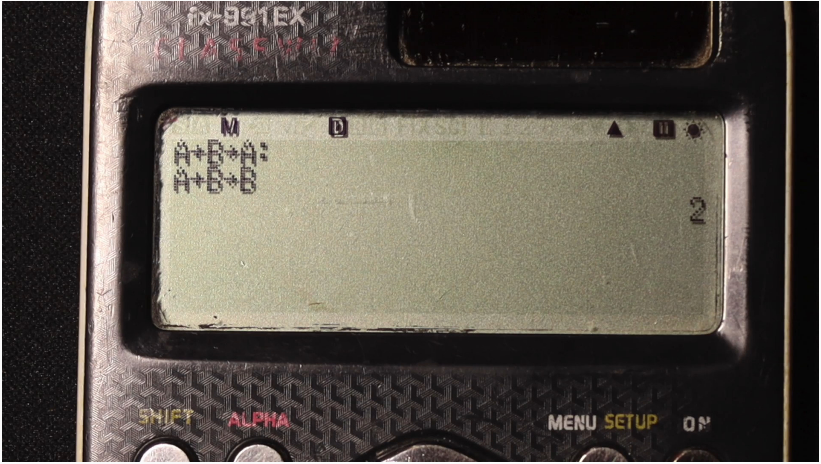
Combining these features



Combining these features



Combining these features



Combining these features

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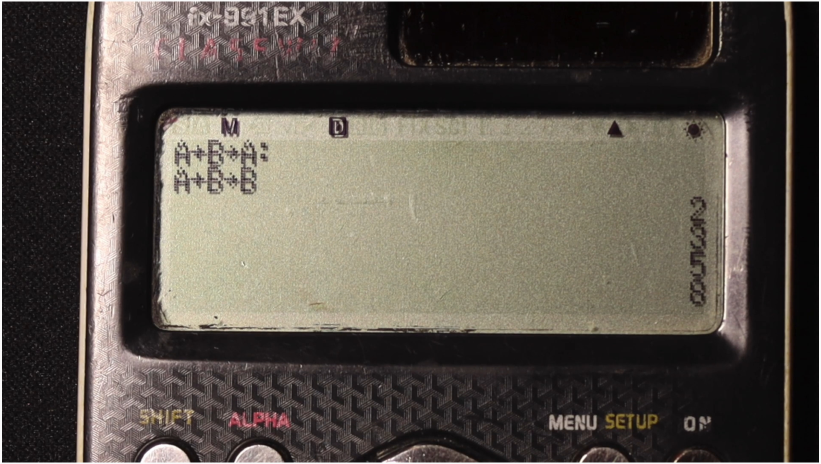


Combining these features

Combining these features

The reduction

Can these two features be combined to simulate an arbitrary Turing machine?

Idea: For any fraction game

$$f_1, f_2, \ldots, f_k$$

and with start value n, find something we can type into the calculator which will simulate all the steps in the corresponding fraction game.

The reduction

This does the trick:

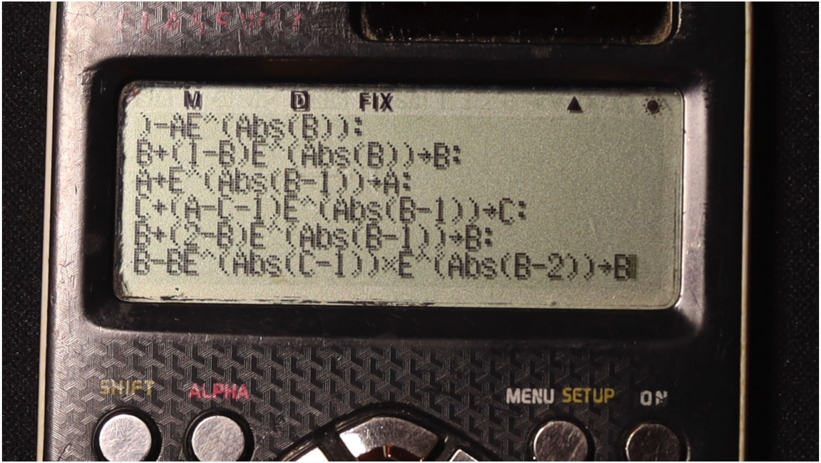
$$\mathsf{n} \to \texttt{initial} :$$
$$\mathsf{n} + (\mathsf{n} \times f_1 - \mathsf{n}) \times \mathsf{delta}(\mathsf{n} \times f_1, \mathsf{floor}(\mathsf{n} \times f_1)) \times \mathsf{delta}(\texttt{initial}, \mathsf{n}) \to \mathsf{n} :$$
$$\mathsf{n} + (\mathsf{n} \times f_2 - \mathsf{n}) \times \mathsf{delta}(\mathsf{n} \times f_2, \mathsf{floor}(\mathsf{n} \times f_2)) \times \mathsf{delta}(\texttt{initial}, \mathsf{n}) \to \mathsf{n} :$$
$$\vdots$$
$$\mathsf{n} + (\mathsf{n} \times f_k - \mathsf{n}) \times \mathsf{delta}(\mathsf{n} \times f_k, \mathsf{floor}(\mathsf{n} \times f_k)) \times \mathsf{delta}(\texttt{initial}, \mathsf{n}) \to \mathsf{n} :$$
$$\mathsf{sqrt}(-\mathsf{delta}(\mathsf{n}, \texttt{initial}))$$

and repeatedly press equals.

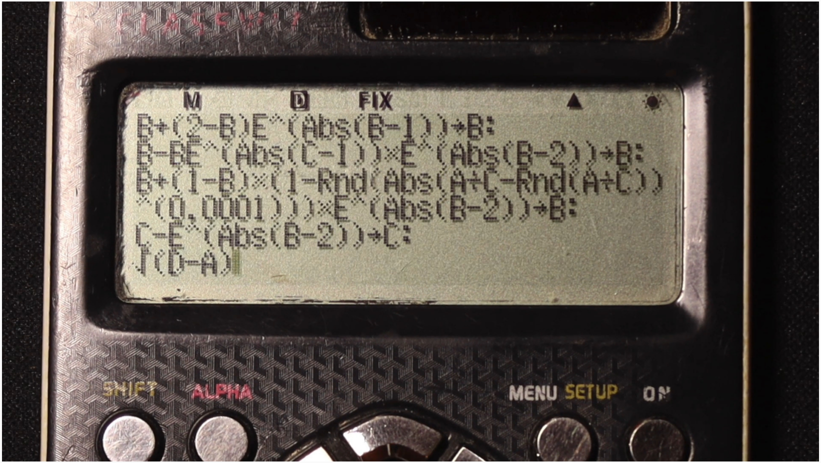
delta

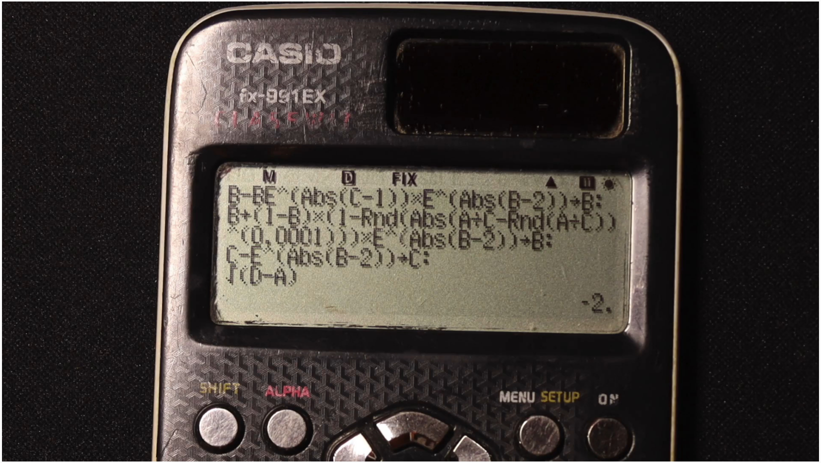
$$\mathsf{delta}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Enumerating primes in an exam



Enumerating primes in an exam




CASIO
fx-991EX
M D FIX
B-BE^(Abs(C-1))×E^(Abs(B-2))→B:
B+(1-B)×(1-Rnd(Abs(A÷C-Rnd(A÷C))
^(0,0001)))×E^(Abs(B-2))→B:
C-E^(Abs(B-2))→C:
∫(D-A)
-2.
SHIFT
ALPHA
MENU SETUP
ON

Enumerating primes in an exam

Enumerating primes in an exam

Practical considerations

Some practical considerations:

- The calculator has a very limited amount of memory
- There are only 10 variables available for you to use
- Writing programs takes a long time if you have to clear your calculator
- The programs are very slow to execute
- Input and output is very limited and confusing
- What if the exam asks instances of undecidable problems?

Conclusion

Actually not very useful in practice... it's really just an excuse to talk about an interesting model of computation.

Questions?

Hilbert's 10th problem

One solution for the exam board would be to exclusively ask questions about deciding whether a Diophantine equation has a solution with all unknowns taking integer values.

$$3x^2 - 2xy - y^2z - 7 = 0$$

This is actually undecidable, so to cheat you'd also need to bring an oracle to the Halting problem into an exam, which is much harder.

Catalogue numbers

Definition
A function $f : \mathbb{N} \to \mathbb{N}$ is a computable function if some Turing machine M, on every input w, halts with just $f(w)$ on its tape. check definition!

c	All defined values of $f_c(n)$
0	none
1	$n \mapsto n$
2	$0 \mapsto 1$
4	$0 \mapsto 2$
$\vdots$	$\vdots$
77	$n \mapsto 0$
$7 \cdot 11^{2^k}$	$n \mapsto k$
$\frac{15}{7} \cdot 1029^{2^{k-1}}$	$n \mapsto n + k$

C_π

$$
\begin{aligned}
&2^{100!} + 2^{\frac{365}{46}101\cdot 100!} + 2^{\frac{29}{161}101^{2}100!} + 2^{\frac{79}{575}101^{3}100!} + 2^{\frac{7}{451}101^{4}100!} \\
&+ 2^{\frac{3159}{413}101^{5}100!} + 2^{\frac{83}{407}101^{6}100!} + 2^{\frac{473}{371}101^{7}100!} + 2^{\frac{638}{355}101^{8}100!} + 2^{\frac{434}{335}101^{9}100!} \\
&+ 2^{\frac{89}{235}101^{10}100!} + 2^{\frac{17}{209}101^{11}100!} + 2^{\frac{79}{122}101^{12}100!} + 2^{\frac{31}{183}101^{13}100!} + 2^{\frac{41}{115}101^{14}100!} \\
&+ 2^{\frac{517}{89}101^{15}100!} + 2^{\frac{111}{83}101^{16}100!} + 2^{\frac{305}{79}101^{17}100!} + 2^{\frac{23}{73}101^{18}100!} + 2^{\frac{73}{71}101^{19}100!} \\
&+ 2^{\frac{61}{67}101^{20}100!} + 2^{\frac{37}{61}101^{21}100!} + 2^{\frac{19}{59}101^{22}100!} + 2^{\frac{89}{57}101^{23}100!} + 2^{\frac{41}{53}101^{24}100!} \\
&+ 2^{\frac{833}{47}101^{25}100!} + 2^{\frac{53}{43}101^{26}100!} + 2^{\frac{86}{41}101^{27}100!} + 2^{\frac{13}{38}101^{28}100!} + 2^{\frac{23}{37}101^{29}100!} \\
&+ 2^{\frac{67}{31}101^{30}100!} + 2^{\frac{71}{29}101^{31}100!} + 2^{\frac{83}{19}101^{32}100!} + 2^{\frac{475}{17}101^{33}100!} + 2^{\frac{59}{13}101^{34}100!} \\
&+ 2^{\frac{41}{3}101^{35}100!} + 2^{\frac{1}{7}101^{36}100!} + 2^{\frac{1}{11}101^{37}100!} + 2^{\frac{1}{1024}101^{38}100!} + 2^{101^{39}100!}
\end{aligned}
$$

$$
3 \times 5^{2^{89\cdot 101!}+2^{90\cdot 101!}} \times 17^{101!-1} \times 23
$$